Taylor's Theorem used to approximate functions.

Polynomial constructed that together, with its derivatives, takes the same value as those of function and polynomial at particular point in domain:

$$p(a) = f(a), \quad p'(a) = f'(a), \quad p''(a) = f''(a) \ldots$$

e.g find polynomial approximation to function $f(x)$ such that

$f(0) = 3$, $f'(0) = 4$, $f''(0) = -10$, $f'''(0) = 12$

3rd derivative so looking for cubic polynomial

$p(x) = a + bx + cx^2 + dx^3$

$p(0) = 3$ so comparing coefficients: $a = 3$

$p'(x) = b + 2cx + 3dx^2$

$p'(0) \longrightarrow b = 4$

$p''(x) = 2c + 6dx \quad \therefore \quad c = -5$

$p'''(x) = 6d \quad \therefore \quad d = 2$

$\therefore$ required approximating polynomial $=$ $\boxed{p(x) = 3 + 4x - 5x^2 + 2x^3}$

for $f(1) = 4$, $f'(1) = 0$ etc.

must look for polynomials in powers of $x - 1$

$p(x) = a + b(x-1) + c(x-1)^2 \ldots$

These polynomial approximations to functions are called Taylor approximations:

$f(x) \approx p_n(x)$

where $p_n(x) = \boxed{f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \frac{(x-a)^n}{n!} f^{(n)}(a)}$

The approximation → exact if we add remainder : $R_n(x)$, which can be expressed in many forms, e.g Lagrange's form :

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a+\theta h) \quad \text{where } h = x-a$$

if we replace $x$ in expansion with $x+a$, ie. $f(x+a)$ :

$$\boxed{f(x+a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \dots \frac{x^n}{n!} f^{(n)}(a) + R_n(x)}$$

Taylor Polynomial expansion of $f(x)$ about $x = a$

as $n \to \infty$, $R_n(x) \to 0$ ∴ removed :

$$\boxed{f(x+a) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(a)}$$

setting $a = 0$ leads to special case :

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)}$$

Maclaurin series expansion